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Equivalencies, Identities, Symmetric Differences, and Congruencies in Orthomodular Lattices

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It is shown that operations of equivalence cannot serve for building algebras which would induce orthomodular lattices as the operations of implication can. Several properties of equivalence operations have been investigated. Distributivity of equivalence terms and several other 3-variable expressions involving equivalence terms have been proved to hold in any orthomodular lattice. Symmetric differences have been shown to reduce to complements of equivalence terms. Some congruence relations related to equivalence operations and symmetric differences have been considered.

KEY WORDS: quantum logic; orthomodular lattice; quantum identity; equivalence relation; congruence relation; symmetric difference.

1. INTRODUCTION

It is well-known that any orthomodular lattice equations and conditions generated with at most two generators have classical and quantum constants, variables, and operations—altogether 96 so-called Beran expressions (Beran, 1985; Megill and Pavičić, 2001, 2002). All quantum constants, variables, and operations are fivefold defined by means of classical ones (Pavičić and Megill, 1999). Also, whenever all classical constants in an orthomodular lattice commute the orthomodular lattice becomes the Boolean algebra and quantum constants, variables, and operations reduce to classical ones.

Classical constants are 0 and 1 (Beran expressions 1 and 96), classical variables are a, b (22,39) and their complements $a^\perp$, $b^\perp$ (58,75). Quantum constants are quantum 0's (17,33,49,65,81) and quantum 1's (16,32,48,64,80). Quantum variables are quantum a (6,38,54,70,86), b (7,23,55,71,87), $a^\perp$ (11,27,43,59,91), and $b^\perp$ (10,26,42,74,90) (Megill and Pavičić, 2001, 2002).

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In this paper we show that the binary operations in an orthomodular lattice can be divided into two groups. A group containing operations which together with complementation can be used to express any other operation, and a group which does not enable this. To the former group belong joins and meets and to the latter operations of equivalence. Both of them again have classical and quantum representatives. Classical meet, $\cup$ and join, $\cap$ with a, b, $a^{\perp}$, $a^{\perp}$ (Beran expressions 2–5 and 92–95) and quantum meets and joins (12–21, 28–37, 44–53, 60–69, and 76–85) from the former group and classical equivalence, $\equiv$, and its complement (88 and 9) and quantum equivalences, $\equiv_i$, $i = 1, \ldots, 5$, and their complements (24,25,40,41,56,57,72,73,8,89) (Megill and Pavičić, 2001, 2002). In the field of quantum logic and orthomodular lattices meet and joins have, in the literature, been given various other names depending on the distribution of complements. For example, implication, conditional, projection, skew operations (Beran, 1985), sharp and flat operations (D'Hooghe and Pykacz, 2000; Pykacz, 2000), etc. Also operations of equivalence and their complements have been given other names like (symmetric) (classical and quantum) identity (Pavičić, 1989, 1993, 1998) and asymmetric (quantum) identities (Megill and Pavičić, 2001, 2002; Pavičić and Megill, 1999) and difference (Beran, 1985; Dorfer *et al*., 1996) and noncommutative symmetric differences (Dorfer, 2002).

In this paper we will concentrate on the equivalence operations and we first show that one cannot use equivalence operations to express other operations and therefore that an equivalence algebra cannot induce orthomodular lattices in a way the implication algebras can (cf. implication algebras given by Abbott, 1976, 1969; Clark, 1973; Georgacarakos, 1980; Hardegree, 1981a,b; Kimble, 1969; Megill and Pavičić, 2003; Pavičić and Megill, 1998b; Piziak, 1974). In Section 3 we give solutions to previously open three-variable problems of expressions containing symmetric equivalence terms. In the end, in Section 4 we prove that recently introduced non-commutative symmetric differences (Dorfer *et al*., 1996) are nothing but complements of asymmetric equivalence relations and that therefore the majority of the results obtained in Dorfer *et al*. (1996) directly follow from the results previously obtained in the literature (Megill and Pavičić, 2001, 2002; Pavičić and Megill, 1999).

2. NO-GO FOR EQUIVALENCE ALGEBRA

All implications from a quantum logic (an orthomodular lattice) reduce to the classical one in a classical theory (a Boolean algebra). So, as we show in Pavičić and Megill (1998a), not only $a \leftrightarrow_i b$ but also $(a \rightarrow_i b) \cap (b \rightarrow_j a)$, $i \neq j$ ($i = 0, 1, \ldots, 5$, where $\rightarrow_i$ correspond to Beran expressions: 94,78,46,30,62,14, respectively) must reduce to $a \leftrightarrow_0 b$ in a classical theory. To handle the Beran expressions, we use programs beran and bercomb. (Megill and Pavičić, 2001). Let us have a look at what we get in an orthomodular lattice in Table I, where B(a, b) are Beran expressions (5 of 96 ones given in Beran (1985).

Table I. Products $(a \to_i b) \cap (b \to_j a)$, $i = 0, \ldots, 5$, $j = 0, \ldots, 5$, Reduced to Beran Expressions

$i_{\downarrow} \backslash j_{\rightarrow}$	$b \to_0 a$	$b \to_1 a$	$b \to_2 a$	$b \to_3 a$	$b \to_4 a$	$b \to_5 a$
$a \to_0 b$	B_{88}(a,b)	B_{56}(a,b)	B_{24}(a,b)	B_{40}(a,b)	B_{72}(a,b)	B_8(a,b)
$a \to_1 b$	B_{72}(a,b)	B_8(a,b)	B_8(a,b)	B_8(a,b)	B_{72}(a,b)	B_8(a,b)
$a \to_2 b$	B_{40}(a,b)	B_8(a,b)	B_8(a,b)	B_{40}(a,b)	B_8(a,b)	B_8(a,b)
$a \to_3 b$	B_{24}(a,b)	B_8(a,b)	B_{24}(a,b)	B_8(a,b)	B_8(a,b)	B_8(a,b)
$a \to_4 b$	B_{56}(a,b)	B_{56}(a,b)	B_8(a,b)	B_8(a,b)	B_8(a,b)	B_8(a,b)
$a \to_5 b$	B_8(a,b)	B_8(a,b)	B_8(a,b)	B_8(a,b)	B_8(a,b)	B_8(a,b)

The expressions $B(i)$, $i = 24,40,56,72$, are asymmetrical and at first we would think it would be inappropriate to name them equivalence operations. But we were able to prove Theorem 2.1 and Theorem 2.2 below and therefore we define symmetric and asymmetric equivalence operations as follows:

$$\begin{aligned}
a \equiv_0 b &\stackrel{\text{def}}{=} (a' \cup b) \cap (a \cup b') && (= B_{88}(a, b)) \\
a \equiv_1 b &\stackrel{\text{def}}{=} (a \cup b') \cap (a' \cup (a \cap b)) && (= B_{72}(a, b)) \\
a \equiv_2 b &\stackrel{\text{def}}{=} (a \cup b') \cap (b \cup (a' \cap b')) && (= B_{40}(a, b)) \\
a \equiv_3 b &\stackrel{\text{def}}{=} (a' \cup b) \cap (a \cup (a' \cap b')) && (= B_{24}(a, b)) \\
a \equiv_4 b &\stackrel{\text{def}}{=} (a' \cup b) \cap (b' \cup (a \cap b)) && (= B_{56}(a, b)) \\
a \equiv_5 b &\stackrel{\text{def}}{=} (a \cup b) \cap (b' \cup a') && (= B_8(a, b)).
\end{aligned}$$

Theorem 2.1. *(Pavičić and Megill, 1999). Ortholattices in which*

$$a \equiv_i b = 1 \quad \Leftrightarrow \quad a = b, \qquad i = 1, \ldots, 5, \tag{1}$$

hold are orthomodular lattices and vice versa.

Theorem 2.2. *(Pavičić, 1998). Ortholattices in which*

$$a \equiv_o b = 1 \quad \Leftrightarrow \quad a = b \tag{2}$$

holds is a Boolean algebra and vice versa.

A natural question which springs from these theorems is whether one can express joins and complements by means of the two above-defined operations of equivalence, i.e., whether "equivalence algebras," analogous to implication algebras (Abbott, 1969, 1976; Megill and Pavičić, 2003; Pavičić and Megill, 1998b), can be formulated. In Pavičić (1998) we answer such a question for the symmetric equivalence, $\equiv_5$, in the negative. By the following theorem we answer to this question in the negative for the classical, $\equiv_0$, and the asymmetric equivalences,

$\equiv_i$, $i = 1, \ldots, 4$, as well. Therewith we also prove that an "equivalence algebra" cannot be formulated.

Theorem 2.3. *Orthocomplementation in an orthomodular lattice can be expressed as* $a' = a \equiv_i 0$, $i = 0, 1, \ldots, 5$. *However, classical and quantum joins (including implications) and classical and quantum meets and their complements in an orthomodular lattice cannot be expressed by means of the operations of equivalence.*

Proof: Free orthomodular lattices with two generators (expressions with two elements) can be represented by the direct product $M_{02} \times 2^4$ (Beran, 1985). Denoting the elements of the Boolean algebra 2^4 by $b_1 = (0, 0, 0, 0)$, $b_2 = (1, 0, 0, 0)$, $\ldots$, $b_{16} = (1, 1, 1, 1)$, we can write down all 96 elements of the lattice in the form (a_i, b_j), $i = 1, \ldots, 6$, $j = 1, \ldots, 16$, where a_i are the elements of the orthomodular lattice M_{02} (also called OM_6; Fig. (1) of Pavičić (1998)). We can easily check that (a_i, b_{12}) through (a_i, b_{15}), $i = 1, \ldots, 6$, are exactly all six joins (quantum and classical; among them, of course, implications), while (a_i, b_2) through (a_i, b_5) are their negations, i.e., quantum and classical meets. When we look at the Boolean part only, we can see that they are all characterized by an odd number of 1's (0's) (either one or three).

Looking at the Boolean parts of the other Beran expressions, we find that they all have an even number of 1's and 0's. Quantum and classical 0's are represented by (0,0,0,0), 1's by (1,1,1,1), x's by (1,1,0,0), $-x$'s by (0,0,1,1), y's by (1,0,1,0), $-y$'s by (0,1,0,1), equivalences by (1,0,0,1), and their negations by (0,1,1,0). Simple checking then shows that whatever expression we introduce into equivalences and/or their negations we always end up with expressions whose Boolean parts have only even number of 1's and 0's. This proves the theorem. □

3. SOME OML EXPRESSIONS CONTAINING EQUIVALENCE TERMS

In Megill and Pavičić (2000) we investigated an equational variety of orthomodular lattices (OMLs) whose equations hold in the lattice of closed subspaces of infinite-dimensional Hilbert space $\mathcal{C}(\mathcal{H})$. We showed that this variety could be defined by an infinite set of symmetry relations for equivalencelike terms. In the variety we were also able to prove a "distributivity of equivalence terms" in Theorem 7.2 of Megill and Pavičić (2001) (we called it the "distributivity of identity terms"), shown as Eq. 6 below. In the two papers we conjectured that this "distributivity" might hold in every OML, but were missing the proof. In the meantime we succeeded in finding one and we provide it below. We also prove several related equations that answer a number of open questions in those papers. All of these results are primarily a consequence of a more general result expressed as Eq. 3 below.

We use the notation $a \equiv b$ as an abbreviation for $a \equiv_5 b$.

Theorem 3.1. *The following equations hold in all OMLs.*

$$(a \to_1 b) \cap (b \to_2 c) \cap (c \to_1 d) \cap (d \to_2 a) = (a \equiv b) \cap (b \equiv c) \cap (c \equiv d) \tag{3}$$

$$(a \to_5 b) \cap (b \to_5 c) \cap (c \to_5 d) \cap (d \to_5 a) = (a \equiv b) \cap (b \equiv c) \cap (c \equiv d) \tag{4}$$

$$(a \to_1 b) \cap (b \to_2 c) \cap (c \to_1 a) \leq a \equiv c \tag{5}$$

$$(a \equiv b) \cap ((b \equiv c) \cup (a \equiv c)) = ((a \equiv b) \cap (b \equiv c) \cup (a \equiv b) \cap (a \equiv c)) \tag{6}$$

$$(a \equiv b) \cap ((b \equiv c) \cup (a \equiv c)) \leq a \equiv c \tag{7}$$

$$(a \equiv b) \to_0 ((a \equiv c)(b \equiv c)) = 1 \tag{8}$$

Proof: For Eq. 3, we have

$$\begin{aligned}
&(a \to_1 b) \cap (b \to_2 c) \cap (c \to_1 d) \cap (d \to_2 a) \\
&\quad = (b \to_2 c) \cap (c \to_1 d) \cap (d \to_2 a) \cap (a \to_1 b) \\
&\quad = ((b' \cap c') \cap (c \cap d)) \cap ((d' \cap a') \cap (a \cap b)) \\
&\quad = (b' \cap c' \cap d' \cap a') \cup (b' \cap c' \cap a \cap b) \\
&\qquad \cup (c \cap d \cap d' \cap a') \cup (c \cap d \cap a \cap b) \\
&\quad = (b' \cap c' \cap d' \cap a') \cup 0 \cup 0 \cup (c \cap d \cap a \cap b) \\
&\quad = (a \equiv b) \cap (b \equiv c) \cap (c \equiv d).
\end{aligned}$$

For the second step we used Lemma 3.14 of Megill and Pavičić (2000). For the third step we used the Marsden–Herman Lemma, given for example, as Corollary 3.3 of Beran (1985, p. 259). For the last step we used Lemma 3.11 of Megill and Pavičić (2000).

Equation 4 follows easily from Eq. 3, noticing that $a \equiv b \leq a \to_5 b \leq a \to_1 b, a \to_2 b$. We twice use the transitive law $(a \equiv b) \cap (b \equiv c) \leq a \equiv c$, which is Theorem 2.8 of Megill and Pavičić (2000), in order to establish

$$(a \equiv b) \cap (b \equiv c) \cap (c \equiv d) = (a \equiv b) \cap (b \equiv c) \cap (c \equiv d) \cap (d \equiv a) \tag{9}$$

for the purpose of the proof.

Equation 5 is obtained from Eq. 3 by substituting a for d, then using in the trivial $a \rightarrow_2 a = 1$ on the left-hand side and symmetry of equivalence $a \equiv c = c \equiv a$ on the right-hand side.

In the proof of Eq. 6 in Theorem 7.2 of Megill and Pavičić (2001), the only use of the (stronger-than-OML) Godowski equations was to establish Eq. 5 above. Since we now have a proof that Eq. 5 holds in all OMLs, it follows that Eq. 6 also holds in all OMLs.

Equations 7 and 8 follow from Eq. 6 by Theorem 2.9 of Megill and Pavičić (2000). □

Now we address some open questions answered by this theorem. In Megill and Pavičić (2001), we wondered if Eq. 5 above holds in all OMLs; the answer is affirmative. In addition, together with Eq. 6 above this result answers all open questions posed in the paragraph after Theorem 3.16 of Megill and Pavičić (2000). Equations. 6, 7, and 8 answer the question, posed after Theorem 2.9 of Megill and Pavičić (2000), of whether these equations hold in all OMLs.

Equation 4 above extends the three-variable version of it, given as Eq. 3.21 of Lemma 3.14 of Megill and Pavičić (2000), to four variables. This in turn allows us to prove the assertion of Theorem 3.15 of that paper for $n = 4$ (although that assertion still remains an open problem for $n > 4$). It is unknown whether Eq. 4 holds in all OMLs when extended to five variables. An extension of Eq. 4 to six (or more) variables does not hold in all OMLs, because it fails in the OML of Fig. 2(a) of Megill and Pavičić (2000). (We mention that the extension of Eq. 4 to any number of variables *does* hold in the lattice $\mathcal{C}(\mathcal{H})$, since it is a consequence of Theorem 3.12 of Megill and Pavičić (2000)).

Recall that a WOML (weakly orthomodular lattice) is an OL in which the following additional condition is satisfied (Pavičić and Megill, 1999):

$$(a' \cap (a \cup b)) \cup b' \cup (a \cap b) = 1 \tag{10}$$

In Megill and Pavičić (2000) we asked whether Eqs. 6 and 8 above hold in all WOMLs. The next theorem provides the answer.

Theorem 3.2. *Equation 8 holds in all WOMLs. Equation 6 does not hold in all WOMLs.*

Proof: Equation 8 holds in all OMLs by Theorem 3.1. Since the left-hand side of Eq. 8 evaluates to 1, it therefore also holds in all WOMLs by Lemma 3.7 of Pavičić and Megill (1999).

Equation 6 fails in the WOML of Fig. 1. □

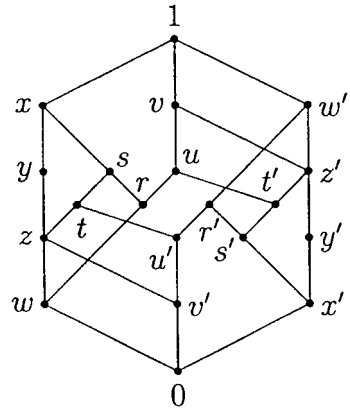

Fig. 1. WOML that violates Eq. 6. (Found by Mike Rose and Kristin Wilkinson at Argonne National Laboratory with the program SEM; Zhang and Zhang (1995)).

4. EQUIVALENCES VS. DIFFERENCES

A recent paper (Dorfer, 2002) "deal[s] with the following question: What is the proper way to introduce symmetric differences in orthomodular lattices? Imposing two natural conditions on. this operation, six possibilities remain." In this section we show that these "six possibilities" are complements of the six equivalence operations from Pavičić and Megill (1999) and Section 2. We also draw the readers' attention to the fact that Navara's technique of handling two variable OML expressions used in Dorfer (2002) have previously been given a computer program support (Megill and Pavičić, 2001) which directly gives all needed results. In the end we comment on the congruence relations from Dorfer (2002) and from Pavičić and Megill (1999).

Below, on the left-hand sides of the equations are symmetric differences from Theorem 2 of Dorfer (2002).

$$
\begin{aligned}
a \nabla b &= (a \equiv_0 b)' = \mathrm{B}_9(a, b) \\
a \Delta b &= (a \equiv_5 b)' = \mathrm{B}_{84}(a, b) \\
a +_l b &= (a \equiv_1 b)' = \mathrm{B}_{25}(a, b) \\
a +_r b &= (a \equiv_4 b)' = \mathrm{B}_{41}(a, b) \\
a +_{l'} b &= (a \equiv_3 b)' = \mathrm{B}_{73}(a, b) \\
a +_{r'} b &= (a \equiv_2 b)' = \mathrm{B}_{57}(a, b)
\end{aligned}
$$

Hence, Definition 1, Theorem 2, most two variable parts of Propositions 3–14, and those of Corollaries 5–13 of Dorfer (2002) directly follow from Pavičić and Megill (1999) (see Section 2) and Megill and Pavičić (2001) (program `beran`). For example, for the proof of Corollary 8 of Dorfer (2002) ($(x +_l y) +_l y = x$) we write

$$\texttt{beran "} - (-(\texttt{x} \equiv_1 \texttt{y}) \equiv_1 \texttt{y})\texttt{"}$$

and we get the output `75 x`, where `75` stands for the Beran expression (x).

In Pavičić and Megill (1999), it was shown that each of $a \equiv_i b = 1$, $i = 0, 1, \ldots, 5$, is a relation of equivalence and of congruence. Therefore, in an ortholattice, OL or in a WOML (see Section 3) there are five such congruence relations. In an orthomodular lattice, OML, due to Theorem 2.1 they all reduce to the following equality: $a = b$. In Dorfer (2002) in Section 3, congruence relations are considered in relation to symmetric differences but are not explicitly defined. For example, in Theorem 15 (iii) of Dorfer (2002) we read: $a\theta b$ iff $(a \equiv_5 b)' \in I$, where I a p-ideal; in Theorem 15 (iii′) we are offered $a\theta b$ iff $(a \equiv_i b)' \in I$. It would be interesting to know examples of $a\theta b$ in orthomodular lattices and which $a\theta b$ relations would satisfy the conditions from the aforementioned Dorfer's Theorem 15 in any orthomodular lattice.

5. CONCLUSION

In Section 2 we show that six operations of equivalence in an orthomodular lattice and the Boolean algebra, we introduced in Pavičić and Megill (1998a), cannot build equivalence algebras which would yield orthomodular lattices in the way the implication algebras.

In Section 3 we show that the distributivity of equivalence terms holds in any orthomodular lattice which has been an open problem so far. We actually prove a more general result, in the form of Eq. 3, that has as a consequence this distributivity as well as the answer to several other open problems raised in previous papers.

In Section 4 we show that six symmetric differences from Dorfer (2002) are nothing but complements of the six equivalence operations from Pavičić and Megill (1999). We also draw the readers' attention to the fact that Navara's technique of handling two-variable OML expressions used in Dorfer (2002) have previously been given a computer program support (Megill and Pavičić, 2001) which directly gives all needed results. In the end we consider congruence relations from Dorfer (2002) and Pavičić and Megill (1999).

All two-variable expressions used in the paper have been given their Beran meaning and numbers.

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